



COMPOSITIONS OF COMPLEMENTS OF GRAPHS

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csaba@cs.elte.hu*Received: 9/12/14, Revised: 11/20/15, Accepted: 4/30/16, Published: 5/16/16***Abstract**

A connected partition of a graph G is a partition of its vertex set such that each induced subgraph is connected. In earlier research it was called a connected composition of a graph. We find a polynomial, the defect polynomial of the graph, that describes the number of connected partitions of complements of graphs with respect to any complete graph. The defect polynomial is calculated for several classes of graphs as cycles or matchings.

1. Introduction

Connected partitions of graphs were introduced in [1] (as graph compositions), with the motivation to find a natural generalization of compositions of integers and partitions of finite sets.

Let $G = (V(G), E(G))$ be an undirected, labeled graph with edge set $E(G)$ and vertex set $V(G)$. A *connected partition* of G is a partition of $V(G)$ into vertex sets of connected induced subgraphs of G . The connected subgraphs $\{G_1, G_2, \dots, G_t\}$

provided by the composition of G are the *parts* of a given composition, with the properties that $\cup_{i=1}^t V(G_i) = V(G)$, and for $i \neq j$, $V(G_i) \cap V(G_j) = \emptyset$. The earlier name composition is a bit misleading, as one can think of compositions of graphs as some kind of operation involving two graphs. The notion connected partition is more descriptive.

Throughout this paper let K_n denote the complete graph on n vertices, P_n the path on n vertices, C_n the cycle on n vertices, and $K_{n,m}$ the bipartite graph on n and m vertices.

It is not difficult to see that connected partitions of the complete graph K_n are in one-to-one correspondence with the partitions of its vertex set [1]. Every subset of the vertices of K_n induces a connected subgraph of K_n (which is also a complete graph), and hence each partition gives a connected partition.

Connected partitions of P_n are obtained in the following way [1]. Any connected subgraph of P_n is also a path on some consecutive vertices of P_n , and the components of a connected partition of P_n consist of paths of cardinality a_i so that $\sum_{i=1}^m a_i = n$. Therefore, the path lengths provide a connected partition of the positive integer n , in other words, the representation of n as an ordered sum of positive integers and vice-versa.

An alternative characterization of connected partitions was given in [2]. For a graph $G = (V(G), E(G))$ the components of the connected partition defined by $E'(G) \subseteq E(G)$ are the connected components of the subgraph $G' = (V(G), E'(G))$.

Let $\Pi_c(G)$ denote the set of connected partitions of a given graph G . The set of connected partitions $\Pi_c(G)$ is a subposet of the partition lattice of $V(G)$, where the ordering is the usual ordering of partitions by inclusion, and $\Pi_c(G)$ is a lattice. The *connected partition number* of G is the number of connected partitions of G , and is denoted by $C(G)$. With these notations, $C(G) = |\Pi_c(G)|$.

The connected partition number of K_n is equal to the number of partitions of its vertex set thus $C(K_n) = \mathcal{B}(n)$, where $\mathcal{B}(n)$ denotes the n th Bell-number. For the path P_n we have $C(P_n) = 2^{n-1}$. Indeed, by omitting an arbitrary subset of the edges of P_n we obtain a connected partition.

Connected partitions of complete bipartite graphs are considered in [3]. The connected partition numbers are found via generating functions. Relationships between flats of matroids and connected partitions are investigated in [4]. A closer relationship between compositions of integers and graphs is analyzed in [5].

In this paper we investigate the connected partition number of complements of graphs. Complements are considered in a more general sense: for a graph G on k vertices we take the complement of G in a complete graph K_n for $n \geq k$. Investigations towards this direction were done in [7]. There, among others, complements of families of graphs, as paths, cycles or stars are considered. We find the connected partition number of complements of several graphs and families of graphs. We also reprove a few of the results of [7].

We introduce the defect polynomial to find the connected partition number of the complement of a graph for every $n \geq k$. A few properties of these polynomials are introduced, and the polynomials are found for some classes of sparse graphs. Furthermore, we give a characterization of the complement of graphs with girth at least five.

2. Complements and Partitions

Throughout the paper complements and relative complements of graphs are investigated. We shall always distinguish the graph from its set of vertices. If no other notation is introduced, the vertices of a graph G are denoted by $V(G)$ and the edges by $E(G)$. Let us introduce the following notation.

Definition 2.1. Let $G = (V(G), E(G))$, and let $H \leq K \leq G$ be subgraphs of G . Then let \overline{H}^G denote the complement of H relative to G , namely $V(\overline{H}^G) = V(G)$ and $E(\overline{H}^G) = E(G) \setminus E(H)$. For a subset of vertices S , where $V(H) \subseteq S \subseteq V(G)$, let \overline{H}^S denote the complement of H with respect to the complete graph on S . Thus $V(\overline{H}^S) = S$ and $E(\overline{H}^S) = \binom{S}{2} \setminus E(H)$. For a subset of vertices $S \subseteq V(H)$ we denote by $H|_S$ the subgraph of H induced on the vertex set S .

The following observation will be used often throughout the paper in inclusion-exclusion principle arguments. The essence of this lemma is implicitly used several times in [7].

Lemma 2.2. *Let $G = (V(G), E(G))$ and $H \leq G$, and let $\mathcal{P} = \{G_1, \dots, G_k\}$ be a connected partition of G . The partition \mathcal{P} does not induce a connected partition on \overline{H}^G if and only if there is a component $G_i \in \mathcal{P}$ such that the complement $\overline{H}|_{V(G_i)}^{V(G_i)}$ is not connected.*

Proof. By definition, an arbitrary partition \mathcal{Q} of $V(G)$ induces a connected partition on \overline{H}^G if and only if for every component $Q_i \in \mathcal{Q}$ the complement $\overline{H}|_{Q_i}^{Q_i}$ is connected. If we assume that the partition \mathcal{Q} is a connected partition, we obtain the statement. \square

Definition 2.3. Let $G = (V(G), E(G))$, and let \mathcal{P} be a partition of G , $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$, such that $P_i \subseteq V(G)$ and $\bigcup P_i = V(G)$. The subsets P_i will be called *parts* or *classes* of the partition \mathcal{P} . A subset P_i is called an *obstacle* of the graph G if it satisfies the conditions of Lemma 2.2, namely that $\overline{G}|_{P_i}^{P_i}$ is not connected. Denote the set of obstacles of G by $\mathcal{A}(G)$.

Observe that an obstacle contains at least two vertices. The obstacles on exactly two vertices are the edges.

A partition \mathcal{P} of $V(G)$ is a connected partition of \overline{G} if and only if \mathcal{P} does not contain an obstacle. This obviously implies that

$$C(\overline{G}^{K_n}) = \mathcal{B}(n) - |\{\mathcal{Q} \mid \mathcal{Q} \in \Pi_c(K_n) \text{ and } \mathcal{Q} \cap \mathcal{A}(G) \neq \emptyset\}|.$$

Recall that $\mathcal{B}(n)$ stands for the n th Bell-number. Note that this is just a reformulation of the definitions.

In order to obtain a useful way to find $C(\overline{G}^{K_n})$, we take a closer look at the partitions with obstacles.

Definition 2.4. Let $\Theta(G)$ be the set of antichains of the obstacles of G :

$$\Theta(G) = \{\mathcal{K} \subseteq \mathcal{A}(G) \mid \text{for all } K_1, K_2 \in \mathcal{K}, \text{ if } K_1 \neq K_2 \text{ then } K_1 \cap K_2 = \emptyset\}.$$

Definition 2.4 describes the subsets of the power set of the obstacles that can be subsets of partitions.

Theorem 2.5. *Let $G \leq K_n$. Then*

$$C(\overline{G}^{K_n}) = \sum_{\mathcal{K} \in \Theta(G)} (-1)^{|\mathcal{K}|} \mathcal{B}(n - |\cup \mathcal{K}|).$$

Proof. By Lemma 2.2 we get that $C(\overline{G}^{K_n})$ is equal to the number of \mathcal{P} connected partitions of K_n such that $\mathcal{P} \cap \mathcal{A}(G) = \emptyset$. The number of connected partitions of K_n is equal to the n th Bell-number, $C(K_n) = \mathcal{B}(n)$. We have to subtract the number of connected partitions that contain an obstacle as a part.

An element $\mathcal{K} \in \Theta(G)$ can be extended to a connected partition of K_n by joining a connected partition of $V(K_n) \setminus \cup \mathcal{K}$. The spanned subgraph is always a complete graph, hence \mathcal{K} can be extended exactly on $\mathcal{B}(n - |\cup \mathcal{K}|)$ ways to a connected partition of K_n .

The inclusion-exclusion principle leads to the statement. □

For the complement of a subgraph $H \leq G$ a similar, but more complicated statement can be formulated.

Definition 2.6. Let $G = (V(G), E(G))$ and $H \leq G$. Let $\mathcal{P} = \{G_1, G_2, \dots, G_k\}$ be a connected partition of G . The subgraph G_i is called an *obstacle of H relative to G* if $\overline{H|_{V(G_i)}}^{V(G_i)}$ is not connected. Observe that $G|_{V(G_i)}$ is connected.

Let $\mathcal{A}^G(H)$ denote the set of relative obstacles of H with respect to G . Let

$$\Theta^G(H) = \{\mathcal{K} \subseteq \mathcal{A}^G(H) \mid \forall K_1, K_2 \in \mathcal{K}, K_1 \neq K_2 \Rightarrow K_1 \cap K_2 = \emptyset\}.$$

Theorem 2.7. *Let $H \leq G$. Then*

$$C(\overline{H}^G) = \sum_{\mathcal{K} \in \Theta^G(H)} (-1)^{|\mathcal{K}|} C(G - \cup \mathcal{K}).$$

Proof. The proof is similar to the proof of Theorem 2.5. By Lemma 2.2, we have to subtract the number of compositions that contain an element from $\mathcal{A}^G(H)$ as a part. However, an element $\mathcal{K} \in \Theta^G(H)$ can be extended to a connected partition of G by joining a connected partition of $G \setminus \cup \mathcal{K}$.

The inclusion-exclusion principle leads to the statement. □

Example 2.8. Let Q_3 be the graph corresponding to the vertices and edges of the 3-dimensional cube. Then $C(Q_3) = 958$.

Proof. Observe that $Q_3 = \overline{H}^{K_{4,4}}$, where H is a matching of $K_{4,4}$; that is, H is composed of four independent edges of $K_{4,4}$. Now, $\mathcal{A}^{K_{4,4}}(H)$ is the set of all stars in $K_{4,4}$ that have at least one common edge with H and all $K_{2,2}$ subgraphs of $K_{4,4}$ which have exactly two common edges with H . By Theorem 2.7 we obtain

$$\begin{aligned} C(Q_3) &= C(K_{4,4}) - 4C(K_{3,3}) - 24C(K_{2,3}) - 24C(K_{1,3}) - 8C(K_{0,4}) \\ &\quad + 48C(K_{1,2}) + 48C(K_{0,2}) + 56C(K_{1,1}) + 48C(K_{0,1}) - 14C(K_{0,0}). \end{aligned}$$

The symbol $K_{0,0}$ denotes the empty graph, and by definition, $C(K_{0,0}) = 1$.

The values $K_{4,4} = 2100$, $K_{3,3} = 128$, $K_{2,3} = 34$, $K_{1,3} = 8$, $K_{1,2} = 4$ and $K_{0,n} = 1$ can be calculated from the formulas in [1] or [3]. □

3. The Defect Polynomial of Graphs

In order to group the Bell-numbers in the expression in Theorem 2.5, let

$$c_i^G = \sum_{\substack{\mathcal{K} \in \Theta(G) \\ |\cup \mathcal{K}|=i}} (-1)^{|\mathcal{K}|}.$$

In this way, if $|V(G)| = k$ we obtain

$$C(\overline{G}^{K_n}) = \sum_{i=0}^k c_i^G \mathcal{B}(n - i). \tag{1}$$

The coefficients c_i^G depend only on $\Theta(G)$; hence they do not depend on n , only on G . It will be shown that there are unique real numbers (integers) c_i^G such that this expression holds for every n .

Lemma 3.1. *Let $G = (V(G), E(G))$ be a graph, $|V(G)| = k$. Let $B(x) = \sum_{n=0}^{\infty} B(n)x^n$ be the ordinary generator function of the Bell numbers, and a_1, \dots, a_k such that for every $n \geq k$*

$$C(\overline{G}^{K_n}) = \sum_{i=0}^k a_i \mathcal{B}(n-i) \tag{2}$$

holds. Then $a_i = c_i^G$ for every $i = 1, 2, \dots, n$. Moreover,

$$B(x)d_G(x) - r(x) = \sum_{n=k}^{\infty} C(\overline{G}^{K_n})x^n \tag{3}$$

for some polynomial $r(x)$ of degree $k - 1$.

Proof. First observe that Formula (3) holds for every k -tuple of numbers satisfying (2). We need only examine the coefficient of x^n in the product for $n > k$. Let $p(x) = \sum_{i=1}^n a_i x^i$ and $q(x) = \sum_{i=0}^k c_i^G x^i$.

Now, by the assumption, there are polynomials $r(x)$ and $r'(x)$ such that

$$B(x)q(x) - r(x) = \sum_{n=k}^{\infty} C(\overline{G}^{K_n})x^n$$

and

$$B(x)p(x) - r'(x) = \sum_{n=k}^{\infty} C(\overline{G}^{K_n})x^n.$$

After subtraction, we get that $B(x)(q(x) - p(x)) - r(x) + r'(x) = 0$, and therefore

$$B(x) = \frac{r(x) - r'(x)}{q(x) - p(x)}. \tag{4}$$

In order to exclude 0 as a pole of the right-hand side, by multiplying by x^n we obtain

$$x^n B(x) = x^n \frac{r(x) - r'(x)}{q(x) - p(x)}. \tag{5}$$

For the coefficients of $B(x)$ we have

$$\limsup \sqrt[n]{B(n)} = \limsup \sqrt[n]{\frac{1}{e} \sum_{k \geq 0} \frac{k^n}{k!}} \geq \limsup \sqrt[n]{\frac{t^n}{t!}} = t.$$

Therefore, $\limsup \sqrt[n]{B(n)} = \infty$.

On the left-hand side of (5) we have a formal power-series with no radius of convergence. On the right-hand side the corresponding expansion has a positive

radius of convergence, unless its denominator is the zero polynomial. Therefore $p(x) = q(x)$ and, also $r(x) = r'(x)$. \square

By Lemma 3.1 the polynomial $q(x)$ is unique and although we substitute Bell numbers for x^i , by Equation (2) it shows some polynomial behavior.

Definition 3.2. The *defect polynomial of G* is

$$d_G(x) = \sum_{i=0}^k c_i^G x^i.$$

If there is no confusion, we write c_i for c_i^G .

Now, we investigate the properties of $d_G(x)$. It really shows some further natural polynomial behaviour.

Lemma 3.3. Let $G = G_1 \cup G_2$ where G_1 and G_2 are two components of G . Then

$$d_G(x) = d_{G_1}(x)d_{G_2}(x).$$

Proof. As there is no vertex of G_1 connected to vertices of G_2 , a class of a partition of G contains either vertices of G_1 or vertices of G_2 , but not both.

The same holds for the obstacles of G : $\mathcal{A}(G) = \mathcal{A}(G_1) \cup \mathcal{A}(G_2)$, and $\mathcal{A}(G_1) \cap \mathcal{A}(G_2) = \emptyset$; moreover, for any $K_i \in \mathcal{A}(G_i)$ for $i = 1, 2$ we have $K_1 \cap K_2 = \emptyset$. So

$$\begin{aligned} d_G(x) &= \sum_{\mathcal{K} \in \Theta(G)} (-1)^{|\mathcal{K}|} x^{|\cup \mathcal{K}|} \stackrel{(1)}{=} \sum_{\mathcal{K}_1 \in \Theta(G_1), \mathcal{K}_2 \in \Theta(G_2)} (-1)^{|\mathcal{K}_1 \cup \mathcal{K}_2|} x^{|(\cup \mathcal{K}_1) \cup (\cup \mathcal{K}_2)|} \\ &\stackrel{(2)}{=} \sum_{\mathcal{K}_1 \in \Theta(G_1), \mathcal{K}_2 \in \Theta(G_2)} (-1)^{|\mathcal{K}_1| + |\mathcal{K}_2|} x^{|\cup \mathcal{K}_1| + |\cup \mathcal{K}_2|} \\ &= \sum_{\mathcal{K}_1 \in \Theta(G_1)} (-1)^{|\mathcal{K}_1|} x^{|\cup \mathcal{K}_1|} \sum_{\mathcal{K}_2 \in \Theta(G_2)} (-1)^{|\mathcal{K}_2|} x^{|\cup \mathcal{K}_2|} \\ &= d_{G_1}(x)d_{G_2}(x). \end{aligned}$$

Equality (1) holds, because the partitions of G split into two classes, a partition of G_1 and a partition of G_2 . Equality (2) is just using the fact that the parts of partitions of G_1 and G_2 are disjoint, hence the sizes are the sum of the sizes of the two parts. \square

Lemma 3.4. Let G be a graph, $v \in V(G)$. Then

$$d_G(x) = d_{G-v}(x) - \sum_{v \in K \in \mathcal{A}(G)} x^{|K|} d_{G-K}(x).$$

Proof. The first term calculates those partitions where v is a single vertex (that is, $\{v\}$ is a class of the partition). The sum handles the remaining ones. \square

Corollary 3.5. *Let kD denote the graph with k independent edges. Then $d_{kD}(x) = (1 - x^2)^k$.*

Proof. Every connected component of kD is a K_2 , and therefore by Lemma 3.3, $d_{kD}(x) = (d_{K_2}(x))^k$. As $d_{K_2}(x) = 1 - x^2$, we obtain the desired result. \square

Example 3.6. Let O denote the vertex-edge graph of the regular octahedron. Then $C(O) = 163$. Indeed, the graph O can be obtained from a complete graph on six vertices omitting three edges. As $d_{3D}(x) = (1 - x^2)^3 = 1 - 3x^2 + 3x^4 - x^6$ we have that $C(O) = \mathcal{B}(6) - 3\mathcal{B}(4) + 3\mathcal{B}(2) - \mathcal{B}(1) = 163$.

Corollary 3.7. *Let $a_k = \sum_{t=1}^n (-1)^t S_2(k, t)$, where $S_2(n, k)$ denotes the 2-associated Stirling number of the second kind. Then*

$$d_{K_n}(x) = 1 + \sum_{k=1}^n \binom{n}{k} a_k x^k.$$

Moreover, let $d(x, y)$ denote the exponential generating function of the polynomials $d_{K_n}(x)$, $d(x, y) = \sum_{n=0}^{\infty} d_{K_n}(x) \frac{y^n}{n!}$. Then

$$d(x, y) = e^{1+y+xy-e^{xy}}.$$

Proof. Observe that every $\mathcal{K} \in \Theta(K_n)$ is a subpartition of $V = V(K_n)$, where every class contains at least two vertices. So we can count them in the following way: for every $1 \leq k \leq n$ we choose k elements from V , then partition it into exactly t subsets, where every member contains at least two elements. So

$$d_{K_n}(x) = 1 + \sum_{k=1}^n \binom{n}{k} \sum_{t=1}^k (-1)^t S_2(k, t) x^k = 1 + \sum_{k=1}^n \binom{n}{k} a_k x^k.$$

Let $F(x, y) = \sum_{n,k>0} S_2(n, k) \frac{x^n}{n!} y^k$. It is known that

$$F(x, y) = e^{y(e^x - 1 - x)} - 1.$$

It follows that $d(x, y) = (F(xy, -1) + 1)e^y$. \square

Corollary 3.8. *For the defect polynomials of paths the following recurrence formula holds: $d_{P_n}(x) = d_{P_{n-1}}(x) - x^2 d_{P_{n-2}}(x) - x^3 d_{P_{n-3}}(x)$.*

Proof. Apply Lemma 3.4 with $H = P_n$ and v the last vertex. Then by omitting v , we obtain a path of length $n - 1$. If the vertex v is contained in an obstacle, then v is either a single edge, or a path of length three. These give the last two terms of the expression. \square

Corollary 3.9. *Let T be a tree and L denote the set of its leaves. Let u be a vertex of T such that $|N(u) \cap L| \geq |N(u)| - 1$, that is every neighbour of u is a leaf except possibly one vertex. Let $L_u = N(u) \cap L$, $l = |N(u) \cap L|$, and let t denote the possible non-leaf neighbour of u . Then*

$$d_T(x) = d_{T-L_u}(x) - x((1+x)^l - 1)(d_{T-u-L_u}(x) + xd_{T-u-L_u-t}(x)).$$

We remark that sometimes the last term does not exist, as the vertex t does not necessarily exist. Hence we define the defective polynomial of the empty graph to be the zero polynomial.

Proof. First of all, observe that every obstacle of the tree is a star on at least three vertices. Indeed, for any path of the tree that is longer than three its relative complement is connected, and complements of stars contain two components, one of them consisting of the single center of the star. Then, dividing the partitions into three parts depending on whether or not they have parts from the star and depending on if they contain t , we have the following:

$$\begin{aligned} d_T(x) &= \sum_{\mathcal{K} \in \Theta(G)} (-1)^{|\mathcal{K}|} x^{|\cup \mathcal{K}|} \\ &= \sum_{\substack{\mathcal{K} \in \Theta(G) \\ L \cap (\cup \mathcal{K}) = \emptyset}} (-1)^{|\mathcal{K}|} x^{|\cup \mathcal{K}|} + \sum_{\substack{\mathcal{K} \in \Theta(G) \\ L \cap (\cup \mathcal{K}) \neq \emptyset \\ t \notin \cup \mathcal{K}}} (-1)^{|\mathcal{K}|} x^{|\cup \mathcal{K}|} + \sum_{\substack{\mathcal{K} \in \Theta(G) \\ L \cap (\cup \mathcal{K}) \neq \emptyset \\ t \in \cup \mathcal{K}}} (-1)^{|\mathcal{K}|} x^{|\cup \mathcal{K}|} \\ &= d_{T-L}(x) - \sum_{\emptyset \neq A \subseteq L} x^{|\cup \{u\}|} d_{T-L-u}(x) d_{L-A}(x) - \\ &\quad - \sum_{\emptyset \neq A \subseteq L} x^{|\cup \{u,t\}|} d_{T-L-u-t}(x) d_{L-A}(x) \\ &= d_{T-L}(x) - \sum_{\emptyset \neq A \subseteq L} x^{|\cup \{u\}|} d_{T-A-u}(x) - \sum_{\emptyset \neq A \subseteq L} x^{|\cup \{u,t\}|} d_{T-A-u-t}(x) \\ &= d_T(x) \\ &= d_{T-L}(x) - x((1+x)^l - 1)d_{T-u-L}(x) - x^2((1+x)^l - 1)d_{T-u-L-t}(x). \end{aligned}$$

\square

Corollary 3.10. *Let $K_{1,k}$ be a star on $k + 1$ vertices. Then $d_{K_{1,k}}(x) = (1+x) - x(1+x)^k$.*

Proof. Use Corollary 3.9 for the case when u is the center of the star. Every subset of the leaves is an empty graph, hence the number of its partitions is one, and the corresponding polynomials are the constant one polynomials. \square

Example 3.11. Apply Corollary 3.10 for the graph K_{n+1} and the star S with center-vertex $u \in V(K_{n+1})$ and all edges containing u . Then $C(\overline{S}) = \mathcal{B}(n)$, and the right-hand side gives $\mathcal{B}(n+1) - \sum \binom{n}{k} \mathcal{B}(n-k)$, giving back the well-known formula

$$\mathcal{B}(n+1) = \sum_0^n \binom{n}{k} \mathcal{B}(n-k).$$

4. Obstacles in Graphs with Girth at Least 5

As we have seen earlier, in trees the only obstacles are the stars. A star in an arbitrary graph G consists of a center $v \in V(G)$ and a subset of vertices of its neighbors, $A \subseteq N(v)$. The subgraph induced by the vertex set $\{v\} \cup A$ will be connected in G . However, in its complement, v will stand as an isolated vertex. So it means that the stars are always in $\mathcal{A}(G)$.

In the next lemma we will prove that if the girth of a graph G is at least five, then these stars are the only obstacles.

Lemma 4.1. *Let G be a graph with girth at least five. Then $\mathcal{A}(G)$ is exactly the set of the stars of G .*

Proof. By the definition of $\mathcal{A}(G)$ every star is part of it. Let $K \subseteq V(G)$ with $|V(K)| \geq 4$, and suppose that $H = G|_K$ is not a star, and \overline{H} is not connected. Then there exist $X, Y \subset V(H)$ such that $|X|, |Y| \geq 2$, $X \cup Y = V(H)$, $X \cap Y = \emptyset$, and there is no edge of \overline{H} between X and Y . This means that for every $x \in X$ and $y \in Y$ the pair (x, y) is in $E(H)$. Therefore, there exists a C_4 in H . But this is a contradiction, because $5 > g(H) = g(G|_K) \geq g(G) \geq 5$. \square

From now on, we can deal with graphs of girth at least five in the convenient way described in Lemma 4.1. For example, we can calculate the connected partition numbers of the Petersen graph and the graph of the dodecahedron, both fulfilling this criterion.

We calculated the two polynomials by computer. Our algorithm was based on the idea of Bodo Lass [6].

Let $n = |V(G)|$ and consider the ring $\mathbb{Z}[x_1, \dots, x_n]/(x_i^2 = 0)$, where the variables correspond to the vertices of G . At first we calculated the polynomial $v(x)$, the sum of the characteristic monomials of the obstacles in the ring. Then we calculated $e^{-v(x)} = \sum_{k=0} \frac{(-v(x))^k}{k!}$ and the coefficients of this expansion gave us the coefficients of the defect polynomials. (Observe that any element in the ring without constant

is a nilpotent element, so the sum contains only finitely many nonzero members, while every obstacle contains at least two vertices.)

Corollary 4.2. *Let P denote the Petersen graph and let $n \geq 10$. Then*

$$C(\overline{P}^{K_n}) = \mathcal{B}(n) - 15\mathcal{B}(n-2) - 30\mathcal{B}(n-3) + 65\mathcal{B}(n-4) + 240\mathcal{B}(n-5) \\ + 80\mathcal{B}(n-6) - 450\mathcal{B}(n-7) - 480\mathcal{B}(n-8) + 10\mathcal{B}(n-9) + 119\mathcal{B}(n-10).$$

Corollary 4.3. *Let D denote the graph of the dodecahedron and let $n \geq 20$. Then*

$$C(\overline{D}^{K_n}) = \mathcal{B}(n) - 30\mathcal{B}(n-2) - 60\mathcal{B}(n-3) + 355\mathcal{B}(n-4) + 1380\mathcal{B}(n-5) \\ - 890\mathcal{B}(n-6) - 12000\mathcal{B}(n-7) - 13595\mathcal{B}(n-8) + 39520\mathcal{B}(n-9) \\ + 109086\mathcal{B}(n-10) + 9900\mathcal{B}(n-11) - 271010\mathcal{B}(n-12) \\ - 303900\mathcal{B}(n-13) + 143390\mathcal{B}(n-14) + 482616\mathcal{B}(n-15) + 226630\mathcal{B}(n-16) \\ - 149080\mathcal{B}(n-17) - 165350\mathcal{B}(n-18) - 38780\mathcal{B}(n-19) + 1061\mathcal{B}(n-20).$$

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